Test 2 / Numerical Mathematics 1 / May 25th 2021, University of Groningen

Instructions

- This document has two pages.
- This examination is *open book*, that is you are allowed to check any material of the course.
- **You have 1 hour and 45 minutes in total to finalize the test and upload your solutions. People with special needs (according to the official information of the Educational support center) have 2 hours in total.**
- Upload the answers on the same sport where you downloaded them.
- The grade will be computed as the number of obtained points, plus 1.
- Do not communicate with other students during the examination.
- Sign the pledge and upload it as a separate file in the same place where you will upload your answers.
- Stay in the online main collaborate room during the examination. Information delivered there is official and part of the instructions of the examination. The activity in the room will be recorded.
- The examination must be written by hand **in a tidy and legible way**, scanned and uploaded as PDF to Nestor. Of course you can also use a tablet to write your answers.
- **Upload the PDF in vertical orientation, such that it requires no rotation to be readable.**
- All answers need to be justified using mathematical arguments.
- Oral checks may be run afterwards, either randomly and/or in case of suspicion of fraud.
- **If you do not follow these instructions you will receive the minimal grade.**

Exercise 1 (6 points)

Consider the functions $f(x) = (2x^2 - 3x - 2)/(x-1)$ and $h(x) = x - 2 + x/(x-1)$. Consider also the sequence defined by $x^{(k+1)} = h(x^{(k)})$, $k \geq 0$, $x^{(0)}$ given.

(a) $\boxed{1.0}$ Show that if x^* is fixed point of $h(x)$, then $f(x^*) = 0$.

(b) $\boxed{1.0}$ Show that not all roots of $f(x)$ are fixed points of $h(x)$.

(c) $\boxed{2.0}$ Show that $h(x)$ is a contraction in a subdomain of $\mathbb{R}$ containing x^*.

(d) $\boxed{0.75}$ Prove that the sequence $x^{(k)}$ converges to one of the roots of $f(x)$ for a starting value $x^{(0)}$ sufficiently close to the root.

(e) $\boxed{0.5}$ Will the sequence converge to different roots of $f(x)$ depending on the starting value $x^{(0)}$? Justify your answer.

(f) $\boxed{0.75}$ Compute the values of p and C such that:

$$\lim_{k\to\infty} \frac{|x^* - x^{(k)}|}{|x^* - x^{(k-1)}|^p} = C$$

Exercise 2 (3 points)

We want to solve the linear system $Ax = b$ for $x \in \mathbb{R}^2$ by using stationary Richardson iterations:

$$x^{(k)} = x^{(k-1)} + \alpha \left(b - Ax^{(k-1)} \right)$$

using as initial guess the vector $x^{(0)} = [1, 0]^\intercal$, $b = [1, 0]^\intercal$. The matrix A is given by:

$$A = \begin{bmatrix} 1+\epsilon & \epsilon \\ -\epsilon & 1-\epsilon \end{bmatrix}, A^{-1} = \begin{bmatrix} 1-\epsilon & -\epsilon \\ \epsilon & 1+\epsilon \end{bmatrix}, \quad 1 > \epsilon > 0.$$

In your answers for (a) and (b), you need to compute each of the expressions and make the calculations, without referring to the results from the lectures or labs.

(a) $\boxed{1}$ Find the value of α that minimizes $\|x - x^{(1)}\|^2$.

(b) $\boxed{1}$ Find the largest possible value of α_* such that $\|x - x^{(1)}\|^2 < \|x - x^{(0)}\|^2$ for all $0 < \alpha < \alpha_*$.

(c) $\boxed{1}$ The results of the previous questions are dependent of the initial guess and the right hand side. If those are unknown a priori, choose the value of α to ensure convergence of the Richardson iterations when $k \to \infty$.